Subdivision as a Fundamental Building Block of Digital Geometry Processing Algorithms

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1 Introduction

Multi media data types such as digital sound, images, and video are now ubiquitous in all areas of computing and daily life. This wide impact was made possible by a number of factors. A key factor in the wide use of a given data type is the ease and economy of acquiring it. Using a rough time line one can observe that this was true for sound in the 70s, images in the 80s, and finally video in the 90s, roughly following the development of computing hardware with its ever increasing cpu and memory resources (Figure 1). Another key factor in the wide use of a given data type is the existence of efficient algorithms for creation, storage, transmission, editing and other manipulations of the data. The mathematical foundation for these algorithms has for a very long time rested on sampling and associated Fourier techniques. Even more recent developments, such as the use of wavelets for image and video compression still rest upon the foundation laid by Fourier analysis. As such, these methods now codified as "Digital Signal Processing" (DSP) have been extraordinarily successful impacting areas ranging from cheap consumer devices such as cell phones and MP3 players to high end scientific computing applications solving some of today's most demanding PDEs, for example.

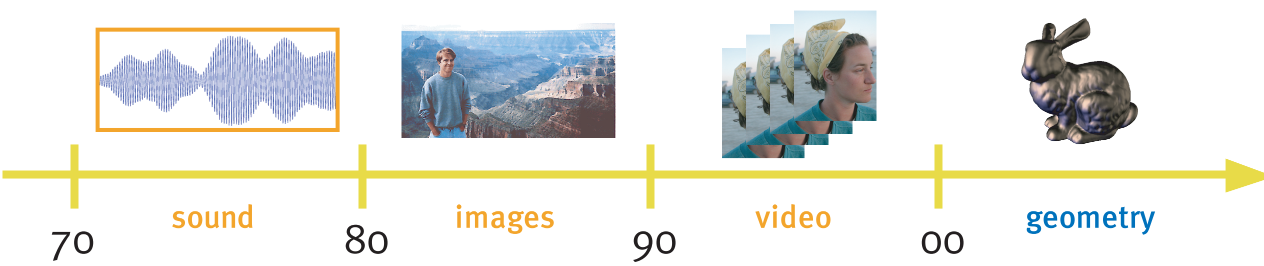


Figure 1: *The development of earlier generations of digital multimedia can be seen as an increase in the dimensionality of the data, enabled by simultaneous advances in acquisition, computation and underlying theory. Digital geometry is now entering a similar phase.*

We are now witnessing the arrival of a new multi media data type: *digital geometry*. As with the earlier waves of multi media this development parallels increasing cpu and memory resources on modern PCs as well as the availability of increasingly cheap and reliable acquisition devices. The latter includes laser range scanners, 3D photography systems based on stereo matching, contact digitizers, as well as volumetric imaging techniques such as industrial and medical MRI and CT scanners (Figure 2). These systems range from low cost consumer level devices all the way to extremely high resolution military and scientific systems. Once again we need corresponding developments in algorithms and computational representations to help us realize the potential this data presents. Unfortunately earlier DSP techniques cannot be immediately brought to bear on this new data type. Instead we need to develop a new toolbox of mathematics, computational representations and algorithms for **Digital Geometry Processing** (DGP) [37].

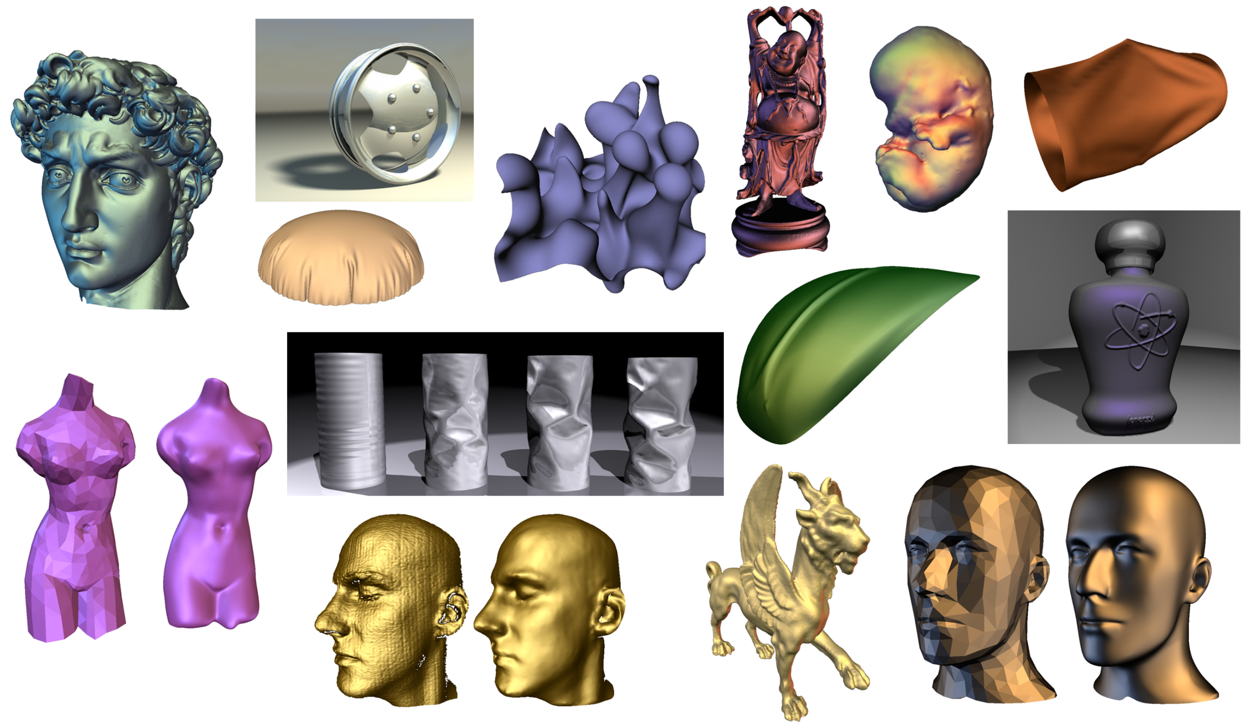


Figure 2: *Examples of digital geometry from many different application areas.*

2 The Need for New Tools

Earlier generations of digital multi media can all be treated as functions of a Euclidian domain. In the case of sound we have a univariate function of time, for images a bivariate function of a section of the image plane, and video may be thought of as a trivariate function of the image plane and time. This is not true anymore for surfaces: They carry intrinsic curvature and their essential 2-manifold (with boundary) structure cannot be ignored. For example, in the Euclidian setting it is possible to use regular sampling to uniquely represent bandlimited functions. There are no comparable uniform, i.e., equispaced, sampling patterns on general 2-manifolds. Since the entire mathematical machinery for digital signal processing is based on translation invariance, none of the DSP tools developed so far are immediately applicable. A possible solution to this problem is the use of local parameterizations on a given 2-manifold for which one could then use regular sampling and apply (windowed) Fourier techniques. However, since there are many possible parameterizations for a given smooth manifold it is not clear which one to use and in general the result of the computation will depend on the particular parameterization chosen. Instead we seek to find sampling patterns for geometry which are as regular as possible (Figure 3).

As an additional ingredient we are also looking for representations which are *hierarchical*. Hierarchical methods in the classical setting add the idea of scale invariance to the already present translation invariance of the Fourier setting and lead to wavelet approaches. Multiresolution of this type, be it called wavelets, or multigrid, or filterbanks, has become a critical ingredient in building highly efficient, scalable algorithms for many processing and numerical computing tasks.

The task then is to rebuild a signal processing like toolbox of mathematical theory and numerical algorithms which replaces regular sampling with *semi-regular* sampling and enables multiresolution algorithms. The latter requires three ingredients: up/down sampling, smoothing, and detail computations. The surface representation, commonly called *Subdivision*, provides these elements and can be used to build the founda-

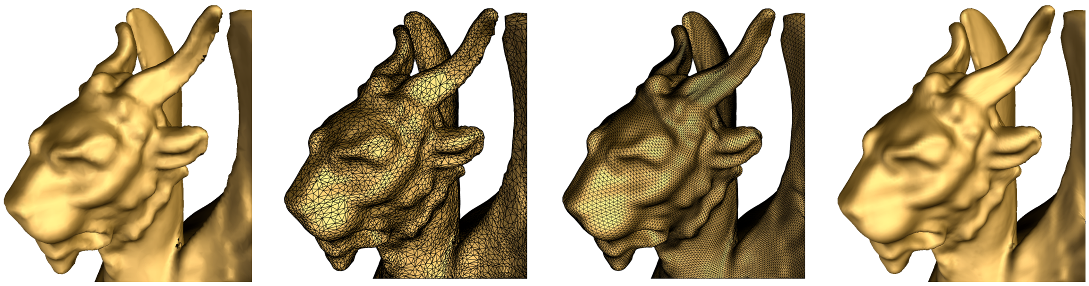


Figure 3: *A surface is typically represented by samples (points) and their connectivity (often triangles). Many sampling technologies result in irregular meshes, i.e., those in which each vertex may have any number of neighbors and triangle sizes do not vary smoothly. Much more attractive are smooth, semi-regular samplings such as the one on the right. There each point has six neighbors almost everywhere and sample spacing varies smoothly. In both cases the geometry itself is the same (leftmost and rightmost).*

tions of Digital Geometry Processing algorithms and theory (Figure 4).

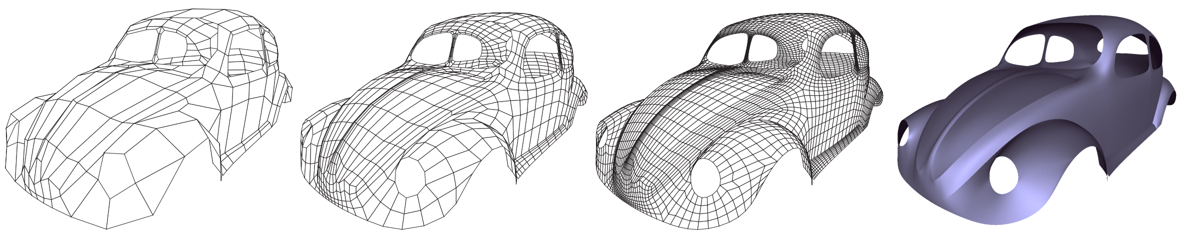


Figure 4: *Subdivision surfaces are constructed through a process of repeated refinement applied to a coarse control mesh. Topologic refinement splits each face in four while the geometric part of refinement assigns spatial positions to all new vertices as well as moves old vertices to new positions. A (piecewise) smooth surface results in the limit. While quadrilaterals are used in this case other schemes based on triangles or even hexagons are possible.*

3 Subdivision Surfaces

Subdivision surfaces were first introduced by Catmull and Clark [4] and Doo and Sabin [12] to address the problem of constructing free-form, smooth surfaces of arbitrary topology. Instead of "glueing" together individual tensor product spline patches, they used the fact that a spline patch can be produced in the limit of repeated uniform knot insertion ("control mesh refinement") and generalized these refinement rules from the regular setting to the arbitrary topology setting (Figure 5).

For quadrilateral schemes vertices with valence $k = 4$ are *regular*—corresponding to the regular quadrangulation of the plane—while those with valence $k \neq 4$ are called *irregular*. In the case of triangle schemes a regular vertex has $k = 6$. The Euler-Poincaré formula for 2-manifolds (with boundary) implies that only a topologic torus, infinite plane, or infinite cylinder admit entirely regular control meshes, indicating that the problem of irregular control points is fundamental and cannot be circumvented.

Early subdivision methods were based on generalization of spline knot insertion [23] and the so-called "Oslo algorithms" [9]. Later on, subdivision was studied in its own right [5]. However, only recently have subdivision surfaces received broad attention and development in the computer graphics community (see [46] and the references therein). In particular, a broad theoretical understanding of the analytic properties of subdivision surfaces was missing for a long time. Early attempts at analysis of the convergence of surface subdivision schemes and the smoothness of the resulting limit functions was based on spectral

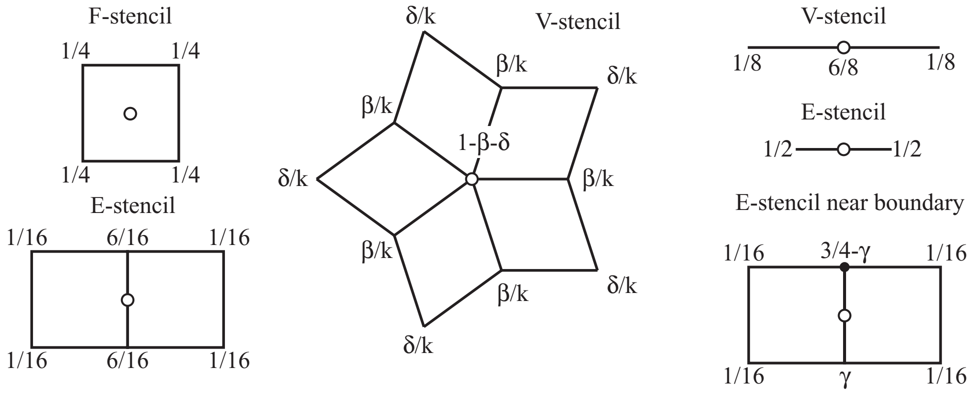


Figure 5: *Refinement rules of Catmull and Clark showing the weights ("stencils") used to compute new control points as averages of control points in the coarser mesh. These rules—face, edge, vertex—are indexed based on the topologic association of the new vertex with the coarser mesh. On the right the boundary rules including one special rule for edges adjacent to a boundary vertex. Here k denotes the number of faces incident on a vertex, $\beta = 3/(2k)$, $\delta = 1/(4k)$, and $\gamma = 3/8 - 1/4\cos(\pi/k)$ where the latter k is taken to be the valence of the boundary vertex. For more details see [3].*

analysis around irregular vertices [2], but was incomplete. Only very recently did this situation change when an essentially complete theoretical understanding of subdivision surfaces was achieved in the works of Reif [34, 36] and Zorin [50, 44, 43].

On the practical side, many recent algorithmic developments have moved applications of subdivision surfaces rapidly forward. Examples include, interactive multiresolution editing [49], reconstruction of sampled data with subdivision surfaces [18], direct evaluation at arbitrary parameter values [38, 45], inclusion of boundary conditions and smoothness constraints [3, 24], trimming [26], approximation with subdivision surfaces [25], simulation of the mechanics of surfaces [7] for engineering design [8], non-manifold subdivision [42], and many more. In fact, techniques are mature enough for deployment in movie production [10] and many geometric modeling packages (e.g., Mirai, Maya, 3DMax, etc.).

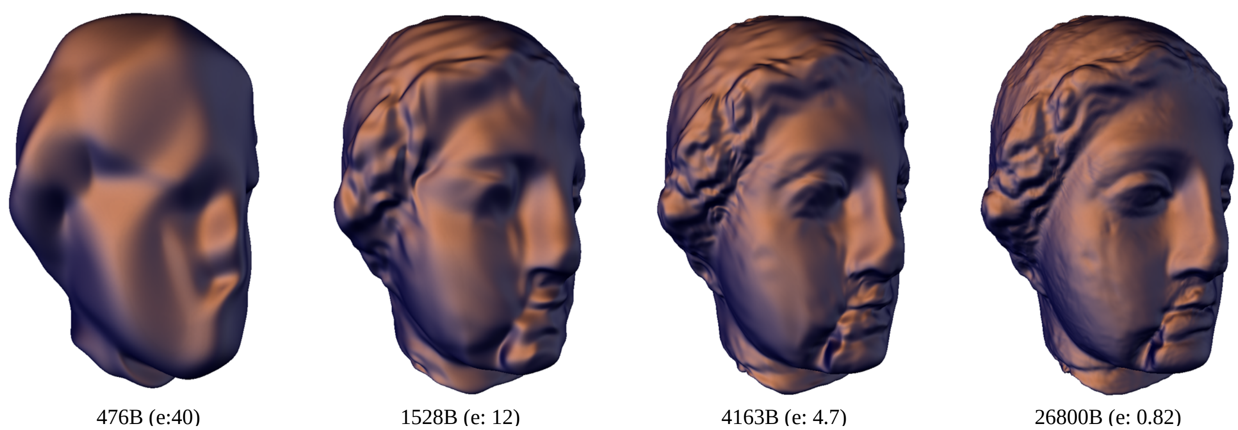


Figure 6: *Partial bit-stream reconstructions from a progressive encoding of the Venus head model. File sizes are given in bytes and relative L^2 reconstruction error in multiples of 10^{-4}. The rightmost reconstruction is indistinguishable from the original.*

A prototypical example of a DGP application enabled by subdivision is the compression and progressive transmission of geometry. Figure 6 illustrates this idea. A surface is encoded into a bitstream which allows partial reconstructions of the original geometry as soon as bits arrive at the receiver. Initially the geometry is “blurry” but improves rapidly. Such constructions are well known from the image setting where wavelets together with zero-tree coders are used for progressive encodings and transmission. Using subdivision basis functions as scaling functions and building associated wavelets [28] one can build compression algorithms for surfaces [19, 6], which are very similar in construction to the image case.

4 The Flavors of Subdivision

As mentioned above, subdivision schemes were originally derived from B-spline knot refinement rules [9] to address the challenge of building smooth surfaces of arbitrary topology [4, 12]. Beginning with an arbitrary connectivity control mesh, which forms a topologic 2-manifold possibly with boundary, the surface is constructed through a limiting process of repeated refinement. Figure 7 shows examples for both both quadrilateral and triangle based schemes, giving the control mesh and the resulting limit surface.

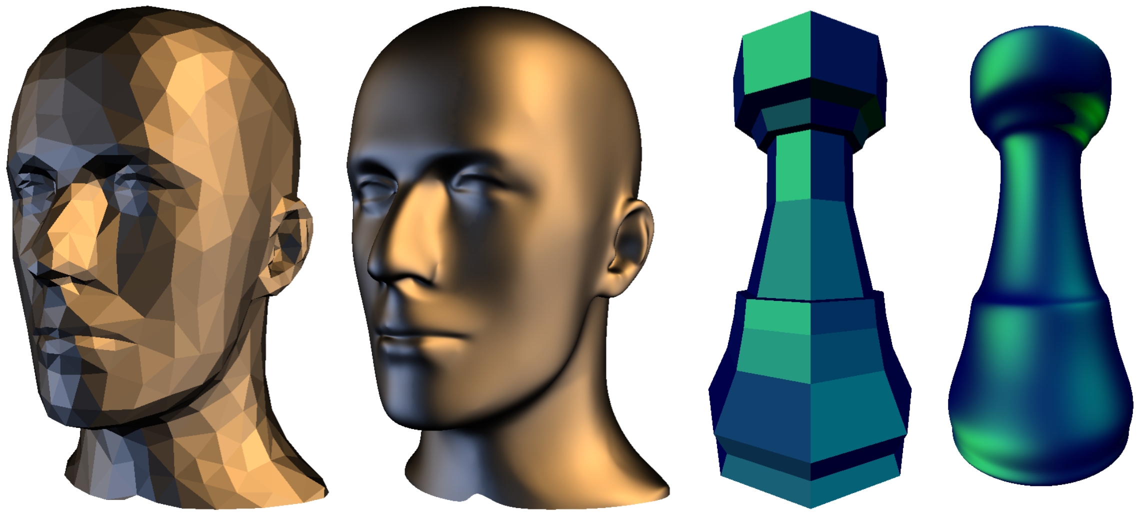


Figure 7: *Examples of subdivision based on triangles and quadrilaterals respectively. On the left, application of Loop’s scheme and on the right, application of the scheme of Catmull and Clark.*

Subdivision consists of two components, a *topologic split* rule describing how the connectivity of the control mesh is refined, and a *geometric rule* which determines the new control point positions from the old positions. For the latter, most constructions use smoothing filters of (small) finite support and local definition with constant coefficients depending only on the valence of vertices or faces in the support of the filter. These schemes can be grouped according to a number of basic criteria (Table 1) indicating whether they are

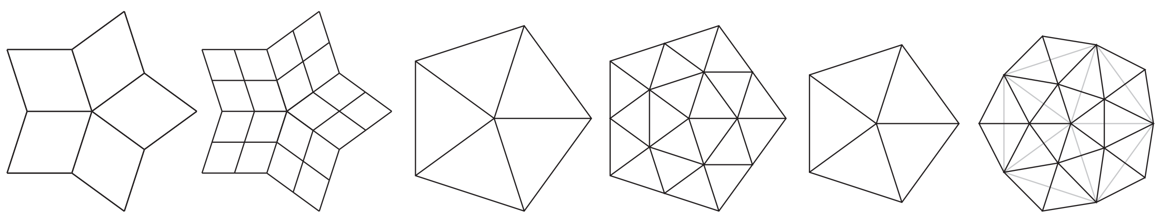
- approximating or interpolating with respect to the original control mesh point positions;
- based on quadrilateral, triangle, or hexagon faces as basic primitive;
- of primal or dual type depending on the split rule (faces or vertices respectively).

Among approximating subdivision schemes, those based on quadrilaterals assume a special role since there exist both primal and dual schemes based on quadrilaterals [12, 4, 17, 32, 40, 41] (in the case of triangles, dual schemes are based on hexagons). Typically the topologic step of a primal scheme is described as a *face split* while dual schemes employ *vertex splits*. Primal schemes which quadrisect are shown in Figure 8(a) for quadrilaterals and Figure 8(b) for triangles. The topologic split step for dual schemes which quadrisect, is illustrated in Figure 8(d) for quadrilaterals and Figure 8(e) for dual triangles, i.e., hexagons.

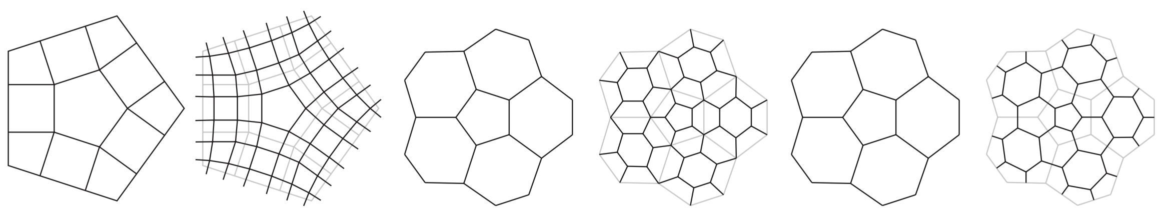
Scheme	approx.	interpol.	quad	triangle	primal	dual
Midedge [17, 32]	*		*			*
Doo-Sabin [12]	*		*			*
Catmull-Clark [4]	*		*		*	
Loop [27]	*			*	*	
Butterfly [14, 48]		*		*	*	
Kobbelt quad. [20]		*	*		*	
Kobbelt $\sqrt{3}$. [21]	*			*	*	
Oswald $\sqrt{3}$. [30]	*			*		*

Table 1: *Classification of major subdivision schemes.*

The Doo-Sabin scheme [12], for example, is a dual quadrilateral scheme, while half box splines [33]—being dual triangle schemes based on vertex quadrisection—lead to hexagonal tilings.



(a) Primal quadrilateral quadrisection. (b) Primal triangle quadrisection. (c) Primal $\sqrt{3}$ (trisection) subdivision.



(d) Dual quadrilateral quadrisection. (e) Dual triangle quadrisection. (f) Dual $\sqrt{3}$ (trisection) subdivision.

Figure 8: *Topologic styles of subdivision. For each pair the coarser level is shown on the left with the finer level on the right.*

Given these basic schemes one may ask for interrelationships between them both in terms of geometric and topologic rules. In the case of univariate splines, for example, the Lane-Riesenfeld algorithm [23] establishes an interlaced ladder of increasing order B-splines through repeated averaging. Here a single geometric rule, which averages back and forth between primal and dual meshes after upsampling, generates B-splines of arbitrary order. For quadrilateral schemes this relationship continues to hold in the arbitrary topology setting [47, 39] (see Figure 9).

Even more interesting are schemes in which the topologic subdivision rule calls for *joining* primal and dual meshes in some fashion. An example are the primal $\sqrt{3}$-schemes [15, 21, 22] in which the refined mesh contains all vertices of the primal and dual mesh (Figure 8(c)). Dual schemes of this type are possible as well (Figure 8(f)). Just as in the case of quadrilateral subdivision, primal and dual $\sqrt{3}$-schemes can also be built based on repeated averaging [30]. Examples of primal $\sqrt{3}$-schemes are shown in Figure 10. Dual schemes, i.e., those based on hexagons and trisection are illustrated in Figure 11.

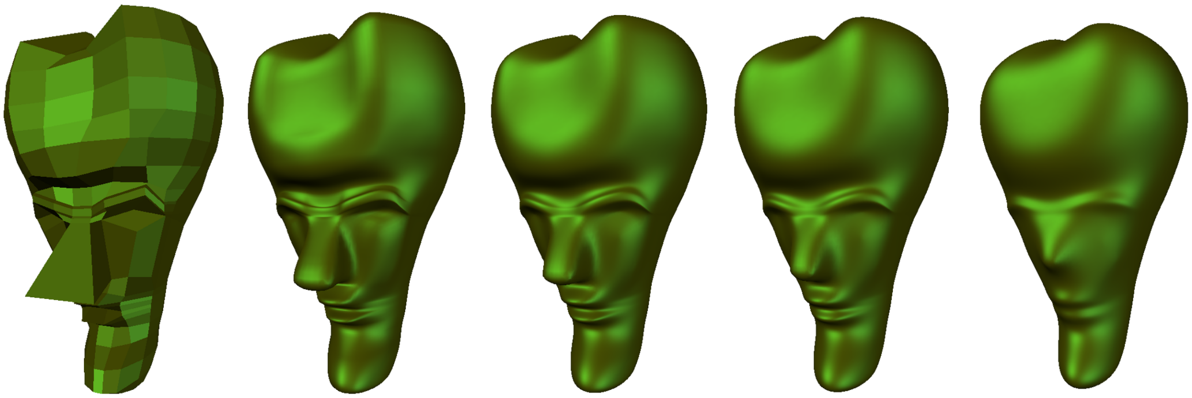


Figure 9: *A more complex example showing the initial control mesh on the left followed by the results of applying subdivision of degree two (Doo-Sabin), three (Catmull-Clark), four, and nine.*

Many more constructions are possible and the area is still quite active. However, in practice Catmull-Clark and Loop subdivision are the most frequently deployed methods.

5 Open Research Questions

The last few years have seen rapid development of fairly comprehensive theory and algorithms for basic surface subdivision, reaching a certain level of maturity. However, the area remains active and fertile for ongoing research. In the following paragraphs we lay out a few of the interesting open questions in the area of subdivision and more generally in Digital Geometry Processing.

Higher order smoothness at irregular points. While C^1 smoothness at irregular points is fairly easy to achieve, no *practical* subdivision schemes which achieve (flexible) C^2 at irregular points are currently known. Lower bound estimates [35] suggest that such schemes with small stencils are unlikely. However, it may be possible to achieve higher smoothness through non-stationary schemes, i.e., schemes which change their stencil (number of coefficients, or coefficient values with fixed stencil) at each level of subdivision.

Unified treatment of subdivision schemes. Repeated averaging which alternates between primal and dual meshes has recently emerged as a general primitive for the construction of broad classes of subdivision. This is true for quadrilateral schemes [39, 47] as well as certain triangle based schemes [39, 30], although the pattern is not as elegant in the latter case. An open question is whether in the case of repeated averaging there exist simple sufficient conditions which ensure the smoothness of the resulting method at irregular points.

Subdivision in higher dimensions. So far subdivision has been mostly employed for curves and surfaces. Many applications domains, especially in physical modeling, require treatment of higher dimensional domains, in particular 3D. Some generalizations of subdivision to 3D [29, 1, 13] and higher dimensions [31] have been pursued, but little is as yet known about them. It appears that the number of cases to distinguish when dealing with the smoothness analysis of such schemes is much larger, considerably complicating the picture in this setting.

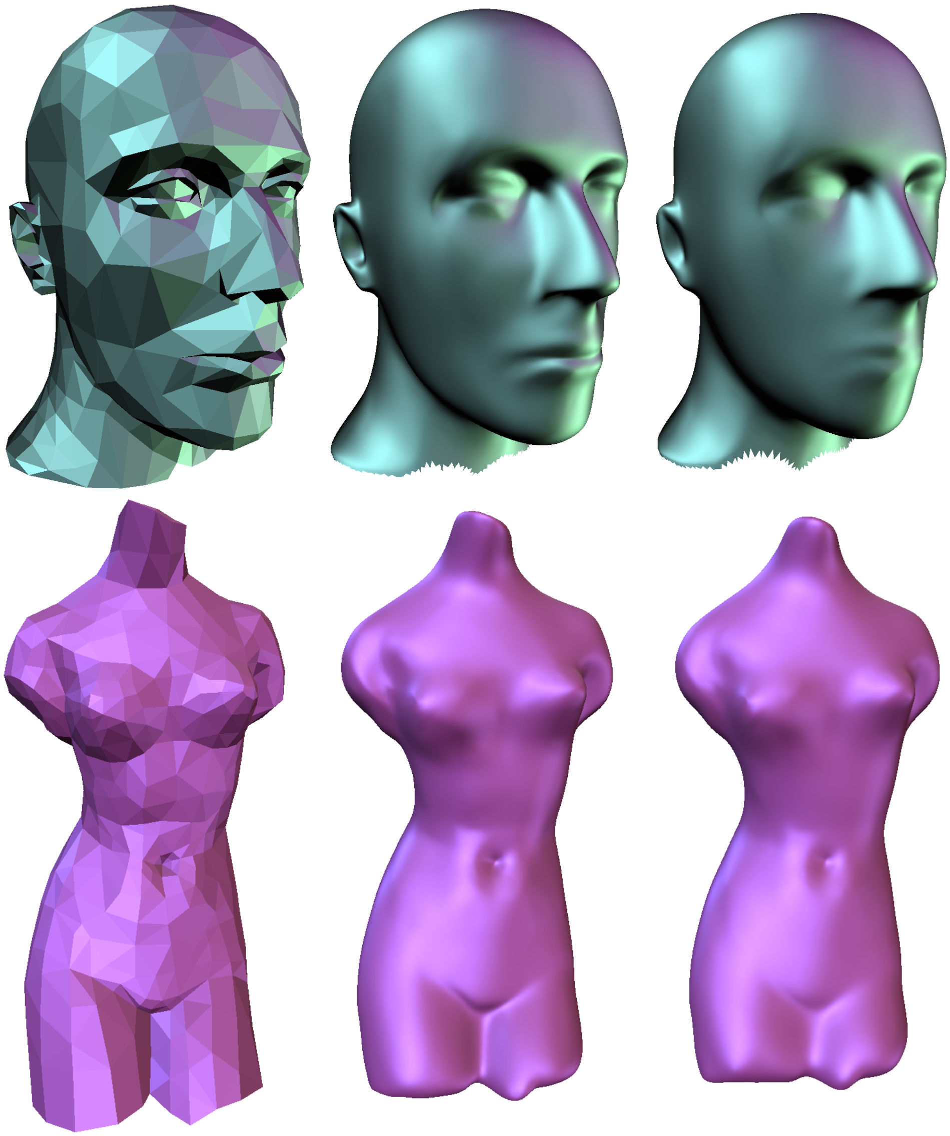


Figure 10: *Examples of repeated averaging based* $\sqrt{3}$ *subdivision with two respectively four averaging cycles for the mannequin and venus models with the respective control meshes on the left.*

Shape insensitive and irregular subdivision. An annoying problem in practical applications of subdivision schemes is the fact that all established schemes are specific to a particular element type. For example, the Loop scheme for triangle meshes and the Catmull-Clark scheme for quadrilaterals. While the latter can be applied to triangle meshes, the resulting surfaces exhibit subtle artifacts such as undulations. In practice it is desirable to work with surfaces which locally have two preferred directions, favoring schemes such as Catmull-Clark. However, when building control meshes for such surfaces it is generally difficult to ensure that they contain quadrilaterals only. Finding a subdivision scheme which performs well, i.e., produces "fair" surfaces independent of the control mesh face types would be highly desirable. A much more general instance of this problem is the design of fully irregular subdivision schemes [16, 11], i.e., those which admit arbitrary control point insertion locations in meshes with irregular connectivity everywhere. Very little is known about such schemes

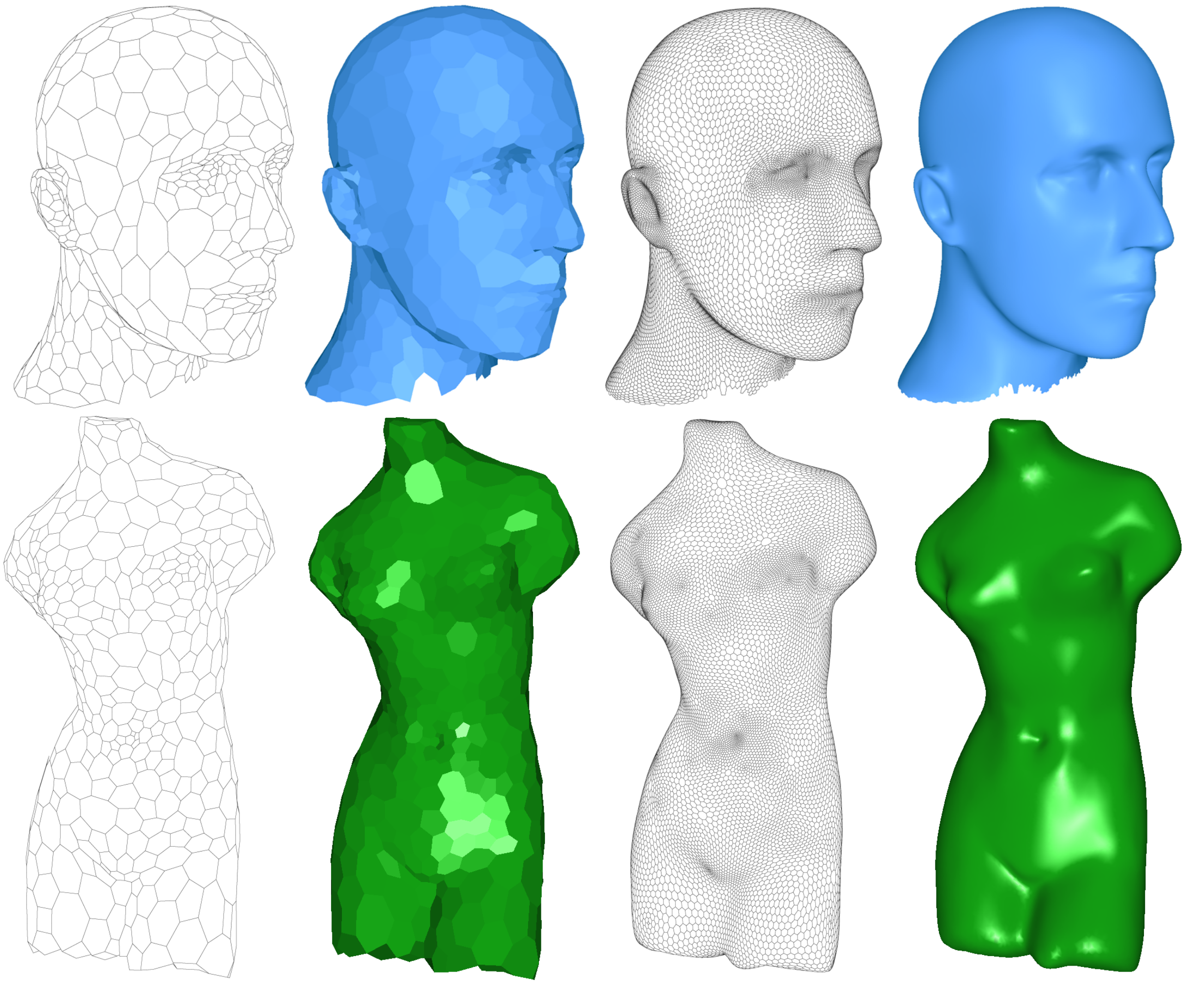


Figure 11: *Dual control mesh and surface after four levels of dual $\sqrt{3}$ subdivision.*

6 Summary

Digital Geometry Processing is a newly emerging research area which aims to make manipulation of geometry as simple and efficient as methods currently in place for image processing, for example. Subdivision surfaces and their underlying machinery and algorithms are a key component to making such algorithms a reality.

Here we aimed to give a short overview of the variety of methods available and provide pointers to the literature for readers interested in more details.

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