A Closed Form Expression for the Form Factor between Two Polygons

Peter Schröder Pat Hanrahan

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Abstract

Form factors are used in computer graphics and radiative heat transfer to describe the fraction of diffusely reflected light or radiation leaving one surface and arriving at another. They are a fundamental geometric property used for computation. Many special configurations admit closed form solutions. However, the important case of the form factor between two polygons in three space has had no known closed form solution. We give a closed form solution for the case of general (planar, convex or concave, possibly containing holes) polygons. The solution is non-elementary since it involves dilogarithms.

1 Introduction

In the analysis and computation of radiative heat transfer and global diffuse illumination in computer graphics the form factor (also referred to as angle factor) plays a central role. It describes the fraction of radiation diffusely emitted from one surface reaching another surface and thus summarizes the geometric relationship between two surfaces in the absence of occlusions. The fast and accurate computation of form factors is of considerable practical import. Textbooks on radiative heat transfer give derivations for many special configurations which admit closed form solutions (see for example [9, 16, 12]). However, up until now the case of the form factor between two general polygons has had no known closed form solution. Since polygons are a very common modeling primitive such a formula would be very useful.

In the first use of radiosity in computer graphics Goral et al. [5] use numerical contour integration to compute form factors between polygons. Cohen and Greenberg [2] used the hemi-cube method to rapidly evaluate form factors using existing computer graphics hardware. Nishita and Nakamae [14] as well as Baum et al. [1] used an exact solution for the form factor between a differential surface element and a polygon to estimate the form factor between finite surfaces. Baum reports greatly reduced errors over earlier methods in particular near singularities. Wallace et al. [18] used ray tracing and closed form expressions to estimate form factors between finite surfaces. Most recently Hanrahan et al. [6] used a hierarchical algorithm which evaluates the form factor integrals adaptively.

The history of computing the amount of light impinging on a diffusely reflecting surface from some light source is very long. A closed form expression for the form factor between a differential surface element and a polygon had already been found by Lambert in 1760 [10]. Lambert proceeded to derive the form factor for a number of special configurations among them the form factor between two rectangles sharing a common edge with an angle of 90 degrees between them. He writes about the latter derivation:

> Although this task appears very simple its solution is considerably more knotted than one would expect. For it would be very easy to write down the differential expression of fourth order, which one would need to integrate four fold; but the highly laborious computation would fill even the most patient with disgust and drive them away from the task. The only simplification which I was able to achieve was to reduce the expression to a second order differential, using the above theorems [formula for differential surface element to polygon form factor], with which I was able to perform the computation.

Lambert also formulates the reciprocity principle in his theorem 16 and uses form factor algebra to compute unknown factors from known ones. The first use of Stokes' theorem [17] to solve for the form factor between two arbitrary surfaces can be found

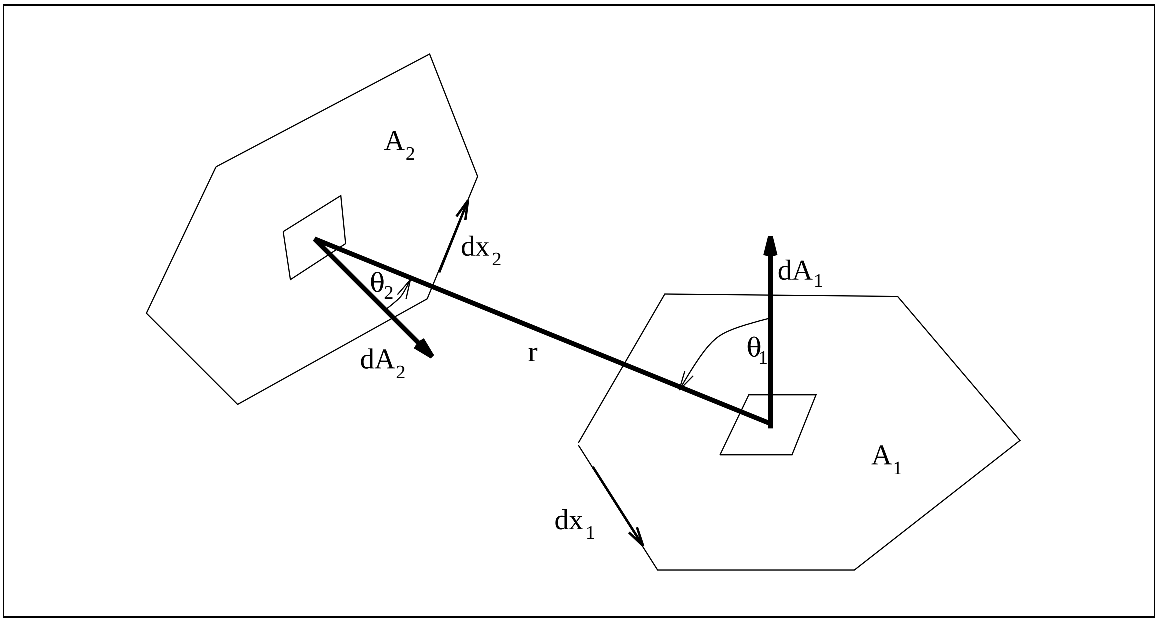

Figure 1: Geometry for the form factor between two surfaces A_1 and A_2. The differential surface elements are shown as $d\vec{A}_1$ and $d\vec{A}_2$, making the angles θ_1 and θ_2 respectively with the connecting vector $\vec{r}$. The boundary elements are shown as $d\vec{x}_1$ and $d\vec{x}_2$.

in a book by Herman in 1900 [7]. Through two applications of Stokes' theorem he reduces the form factor between two arbitrary surfaces to a double contour integral. He uses this result to give the form factor for two parallel quadrilaterals in an exercise. A similar derivation can be found in an article by Fock in 1924 [3]. Fock proceeds by applying the formulation to elliptical disks for which he derives a closed form solution. In 1936 Moon [13], aware of Fock's work, derives closed form solutions for a number of specialized configurations. In the same year, Gershun [4] puts various photometric quantities on a vector calculus footing and gives an especially elegant derivation of the double contour integration using differential forms. More recently Sparrow [15] in 1963 used the double contour form to derive closed form solutions for the case of parallel disks and parallel quadrilaterals. However, none of these sources, or any since, that we are aware of, has given a closed form solution of the form factor between two general polygons.

In this paper we derive a closed form solution for the form factor between two general polygons. In Part 2 we give a short review of the well known double contour integration formula for form factors which is the basis of our closed form. This is followed in Part 3 by a detailed derivation of the symbolic integration of the double contour integral. In Part 4 we apply the method to a simple configuration as an example of its use. The Appendix lists all expressions which need to be computed to give the final answer.

2 Form factors via contour integration

Today the double contour integration method is a standard tool which is derived and taught in many textbooks (e.g. Hottel and Sarofim [8], Love [12], Sparrow and Cess [16]). In the derivation below we will closely follow the treatment of Gershun [4] and Love [12].

The form factor describes the geometric relationship between two surfaces and is defined by

$$\pi A_1 F_{12} = \int_{A_1}\int_{A_2} \frac{\cos\theta_1\cos\theta_2}{\|\vec{r}_{12}\|^2}\, dA_2\, dA_1 \tag{1}$$

where $\theta_{1,2}$ are the angles between the normal vector of the respective surface and a radius vector, $\vec{r}_{12} = -\vec{r}_{21}$, which goes from a point on surface 1 to a point on surface 2 (see Figure 1). The notation A_1 is used for both the surface 1 and its area. Noting that

$$\begin{aligned}\frac{\cos\theta_1\cos\theta_2}{\|\vec{r}_{12}\|^2}\, dA_2\, dA_1 &= -\frac{(\vec{r}_{12}\cdot d\vec{A}_1)(\vec{r}_{12}\cdot d\vec{A}_2)}{\|\vec{r}_{12}\|^4} = -\frac{(\vec{r}_{12}\cdot d\vec{A}_1)\vec{r}_{12}}{\|\vec{r}_{12}\|^4}\cdot d\vec{A}_2 \\ &= \frac{1}{2}\nabla\times\frac{\vec{r}_{12}\times d\vec{A}_1}{\|\vec{r}_{12}\|^2}\cdot d\vec{A}_2\end{aligned}$$

we can at once apply Stokes' theorem, $\int_A(\nabla\times\vec{F})\cdot d\vec{A} = \int_{\partial A}\vec{F}\cdot d\vec{s}$, to Equation 1

$$\pi A_1 F_{12} = \frac{1}{2}\int_{A_1}\int_{\partial A_2}\frac{\vec{r}_{12}\times d\vec{A}_1}{\|\vec{r}_{12}\|^2}\cdot d\vec{x}_2 = \frac{1}{2}\int_{A_1}\int_{\partial A_2}\frac{\vec{r}_{21}\times d\vec{x}_2}{\|\vec{r}_{21}\|^2}\cdot d\vec{A}_1 \tag{2}$$

Given the latter form, we may immediately apply Stokes' theorem again since

$$\frac{\vec{r}_{21}\times d\vec{x}_2}{\|\vec{r}_{21}\|^2} = \nabla\ln\|\vec{r}_{21}\|\times d\vec{x}_2 = \nabla\times(\ln\|\vec{r}_{21}\|\, d\vec{x}_2)$$

giving

$$\begin{aligned}\pi A_1 F_{12} &= \frac{1}{2}\int_{A_1}\int_{\partial A_2}\nabla\times(\ln\|\vec{r}_{21}\|\, d\vec{x}_2)\cdot d\vec{A}_1 \\ &= \frac{1}{2}\int_{\partial A_1}\int_{\partial A_2}\ln\|\vec{r}_{21}\|\, d\vec{x}_2\cdot d\vec{x}_1 \\ &= \frac{1}{4}\int_{\partial A_1}\int_{\partial A_2}\ln(\vec{r}\cdot\vec{r})\, d\vec{x}_2\cdot d\vec{x}_1 \end{aligned} \tag{3}$$

where we dropped the subscript on $\vec{r}_{21}$ since we only need its magnitude.

In the above derivation we have not yet taken advantage of the fact that we only want to consider polygons. A classic result already known to Lambert [10] gives the form factor between a differential surface element and a polygon and follows directly from Equation 2

$$\pi F_{d12} = \frac{1}{2}\int_{\partial A_2}\frac{\vec{r}_{21}\times d\vec{x}_2}{\|\vec{r}_{21}\|^2}\cdot d\vec{A}_1 = \frac{1}{2}\sum_{E_i\in\partial P_2}\vec{\gamma}_i\cdot d\vec{A}_1$$

where $\vec{\gamma}_i$ is a vector normal to the plane spanned by E_i and the location of $d\vec{A}_1$ with magnitude given by the angle subtended by E_i. Baum et al. [1] used the latter form to decrease the error in their radiosity algorithm particularly near singularities. In the case of the form factor between two finite areas (Equation 3) we find for two polygons P_1 and P_2 the expression $d\vec{x}_2 \cdot d\vec{x}_1$ to be a constant for each pair of edges from the two polygons. In this way the total integral decomposes into a sum over all pairwise combinations of edges

$$\pi A_{P_1} F_{P_1 P_2} = \frac{1}{4} \sum_{\substack{E_i \in \partial P_1 \\ E_j \in \partial P_2}} \int_{E_i} \int_{E_j} \ln(\vec{r} \cdot \vec{r})\, d\vec{E}_j \cdot d\vec{E}_i$$

Each one of the latter integrals expands as

$$\int_{E_i} \int_{E_j} \ln(\vec{r} \cdot \vec{r})\, d\vec{E}_j \cdot d\vec{E}_i = \cos \angle E_i E_j \int_0^{l_i} \int_0^{l_j} \ln\left(\left[\vec{x}_i(t) - \vec{x}_j(s)\right] \cdot \left[\vec{x}_i(t) - \vec{x}_j(s)\right] \right) ds\, dt$$

where $l_{i,j}$ are the lengths of $E_{i,j}$ respectively and $\vec{x}_i(t)$ and $\vec{x}_j(s)$ are parameterizations of the edges.

3 Closed form solution

As we have seen in the previous section all that is needed now is to give a closed form solution for integrals of the general form

$$\int_0^{c_2} \int_0^{c_0} \ln f(s,t)\, ds\, dt \tag{4}$$

where c_0 and c_2 are the lengths of the edges over which the given double contour integral is taken (for a complete listing of all constants introduced in the following derivation refer to the Appendix). The bi-quadratic form which arises from the expansion of the dot product is given by

$$\begin{aligned} f(s,t) &= s^2 + c_1 st + t^2 + c_3 s + c_4 t + c_5 \\ &= s^2 + (c_1 t + c_3)s + (t^2 + c_4 t + c_5) \\ &= s^2 + c_6(t)s + c_7(t) \end{aligned}$$

with $c_5 \geq 0$. For later use we also introduce the symbol

$$f_s(s,t) = \frac{d}{ds} f(s,t) = 2s + (c_1 t + c_3)$$

A special case arises immediately if the two line segments under consideration actually lie in a plane. In this case $f(s,t)$ can be factored into two bi-linear forms

$$\begin{aligned} f(s,t) &= (e^{i\theta} s + e^{i\phi} t + \sqrt{c_5})(e^{-i\theta} s + e^{-i\phi} t + \sqrt{c_5}) \\ &= h(s,t)\bar{h}(s,t) \end{aligned}$$

where i is the imaginary unit and θ, ϕ are suitably chosen angles. That this decomposition is possible is most easily seen by considering the two lines in question in a specially chosen coordinate frame. Let both lines be in the s-t plane with one of the lines starting at the origin parallel to the s axis. In this frame, the dot product measuring the length of $\vec{r}$ is given by

$$\begin{aligned}\left\| \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} a + t\cos\psi \\ b + t\sin\psi \\ 0 \end{pmatrix} \right\|^2 &= (s - (a + t\cos\psi))^2 + (b + t\sin\psi)^2 \\ &= (s - e^{i\psi}t - (a - ib))(s - e^{-i\psi}t - (a + ib))\end{aligned}$$

with $a^2 + b^2 = c_5$ and ψ giving the slope of the second line.

Given the above decomposition of f into two factors h and $\bar{h}$, the integral of Equation 4 can be written as

$$\begin{aligned}\int_0^{c_2} \int_0^{c_0} \ln f(s,t)\, ds\, dt &= 2Re\left(\int_0^{c_2} \int_0^{c_0} \ln h(s,t)\, ds\, dt\right) \\ &= Re\left(\frac{h(s,t)^2}{e^{i(\theta+\phi)}}\left(\ln h(s,t) - \frac{3}{2}\right)\right)\Bigg|_{s=0,t=0}^{s=c_0,t=c_2}\end{aligned}$$

where $Re()$ denotes the real part of a complex number.

From now on we will assume that the lines do not share a plane. In particular this implies that they do not intersect. Since $f(s,t)$ measures the distance between the two lines we have $f(s,t) > 0$ for all $(s,t) \in \mathbf{R}^2$. For any fixed but arbitrary value of s the quadratic polynomial $f(s,t)$ (in t) will have a pair of complex conjugate roots with non-zero imaginary part.

For later reference we first consider the following integrals. Let $q(x) = ax^2 + bx + c$ and $d = \sqrt{4ac - b^2}$, then

$$\begin{aligned}\int^y \ln q(x)\, dx &= \frac{q'(y)}{2a} \ln q(y) - 2y + \frac{d}{a} \tan^{-1} \frac{q'(y)}{d} + \kappa_G \\ &=: G(q)(y) &(5)\\ \int^y x \ln q(x)\, dx &= \left(\frac{y^2}{2} + \frac{c}{2a} - \frac{b^2}{4a^2}\right) \ln q(y) \\ &\quad - \frac{y(ay - b)}{2a} - \frac{bd}{2a^2} \tan^{-1} \frac{q'(y)}{d} + \kappa_H \\ &=: H(q)(y) &(6)\end{aligned}$$

for some integration constants $\kappa_{G,H}$. Notice that for polynomials q with non-real roots, d will be real.

We first evaluate the integral with respect to s using Equation 5 setting $\rho(t) = \sqrt{4c_7(t) - c_6^2(t)} \in \mathbf{R}$

$$\int_0^{c_0} \ln f(s,t)\, ds = G(f(.,t))(s)\Big|_{s=0}^{s=c_0}$$

$$= \quad \frac{f_s(s,t)}{2} \ln f(s,t) - 2s + \rho(t) \tan^{-1} \frac{f_s(s,t)}{\rho(t)} \bigg|_{s=0}^{s=c_0} \tag{7}$$

Continuing with the integration of the first two terms on the right hand side of Equation 7—using the integrals in Equation 5 and 6—with respect to t we arrive at

$$\begin{aligned} & \int_0^{c_2} \left(\frac{f_s(s,t)}{2} \ln f(s,t) - 2s \right) \bigg|_{s=0}^{s=c_0} dt \\ = \quad & \left(s + \frac{c_3}{2}\right) G(f(s,.))(t) + \frac{c_1}{2} H(f(s,.))(t) \bigg|_{s=0,t=0}^{s=c_0,t=c_2} - 2c_0c_2 \end{aligned}$$

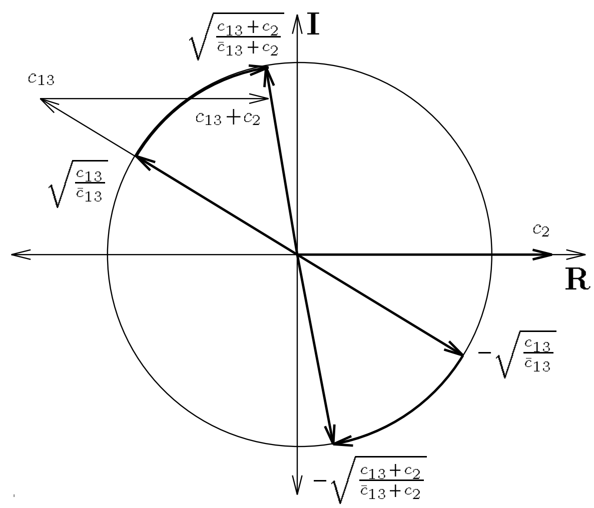

Figure 2: Geometry of the change of variable $t = \frac{\bar{c}_{13}x^2 - c_{13}}{1-x^2}$, which maps the point $t = 0$ onto $x = \sqrt{\frac{c_{13}}{\bar{c}_{13}}}$ and the point $t = c_2$ onto $x = \sqrt{\frac{c_{13}+c_2}{\bar{c}_{13}+c_2}}$. In the figure c_{13} is assumed to be in the upper complex half plane. If it is in the lower half plane the geometry is simply mirrored along the **R** axis. The crucial observation which allows the above construction is that $Im\,(c_{13}) \neq 0$ and $c_2 > 0$.

To integrate the remaining term of Equation 7 with respect to t we first perform a change of variable. Consider the term $\rho(t)$, which we may rewrite as follows

$$\begin{aligned} \rho(t) &= \sqrt{4c_7(t) - c_6^2(t)} \\ &= \sqrt{4(t^2 + c_4 t + c_5) - c_1^2 t^2 - c_1 c_3 t - c_3^2} \\ &= \sqrt{(4 - c_1^2)t^2 + (4c_4 - c_1 c_3)t + (4c_5 - c_3^2)} \\ &= \sqrt{c_{10}t^2 + c_{11}t + c_{12}} = \sqrt{c_{10}(t + c_{13})(t + \bar{c}_{13})} \end{aligned}$$

with $c_{13} = \frac{c_{11} - \sqrt{c_{11}^2 - 4c_{10}c_{12}}}{2c_{10}} = \frac{c_{11}}{2c_{10}} - \frac{c_{14}}{2}$, since $\rho(t) > 0$. The latter also implies that $c_{10} > 0$ and c_{14} purely imaginary. Performing the change of variable $t = \frac{\bar{c}_{13}x^2 - c_{13}}{1-x^2}$ we get

$$
\begin{aligned}
\rho(x) &= \frac{\sqrt{c_{10}}(\bar{c}_{13} - c_{13})x}{1-x^2} = \frac{\sqrt{c_{10}}c_{14}x}{1-x^2} = \frac{c_{15}x}{1-x^2} \\
dt &= \frac{2c_{14}x}{(1-x^2)^2}\,dx \\
\frac{f_s(s,x)}{\rho(x)} &= \frac{(c_1\bar{c}_{13} - c_3 - 2s)x^2 - (c_1c_{13} - c_3 - 2s)}{c_{15}x} \\
&= \frac{\bar{c}_{16}(s)x^2 - c_{16}(s)}{c_{15}x}
\end{aligned}
$$

Since the inverse function of our substitution (square root) is multivalued, care must be taken to use the correct endpoints for the integration path. The substitution projects the real line segment $[0, c_2]$ onto a segment of the unit circle (see Figure 2). The correct endpoints are those which are in the same half plane of the complex plane (either positive or negative imaginary part). Below we use the positive square root for concreteness. Notice that c_{15} is purely imaginary. With this substitution the integral of the last term on the right hand side of Equation 7 becomes

$$
\begin{aligned}
&\int_0^{c_2} \rho(t)\tan^{-1}\frac{f_s(s,t)}{\rho(t)}\bigg|_{s=0}^{s=c_0} dt \\
&= \quad 2c_{14}c_{15}\int_{\sqrt{\frac{c_{13}}{\bar{c}_{13}}}}^{\sqrt{\frac{c_{13}+c_2}{\bar{c}_{13}+c_2}}} \frac{t^2}{(1-t^2)^3}\tan^{-1}\frac{\bar{c}_{16}(s)t^2 - c_{16}(s)}{c_{15}t}\Bigg|_{s=0}^{s=c_0} dt \qquad (8)
\end{aligned}
$$

Before proceeding we take advantage of the fact that $\tan^{-1} z = \frac{1}{2i}\ln\frac{1+iz}{1-iz} + k\pi$ for some integer $k \in \mathbf{Z}$ to yield

$$
\begin{aligned}
\tan^{-1}\frac{\bar{c}_{16}(s)t^2 - c_{16}(s)}{c_{15}t} &= \frac{1}{2i}\ln\frac{c_{15}t - \overline{i c}_{16}(s)t^2 - ic_{16}(s)}{c_{15}t + \overline{i c}_{16}(s)t^2 + ic_{16}(s)} + k(s)\pi \\
&= \frac{1}{2i}\ln\frac{(t - c_{17}(s))(t - c_{18}(s))}{(t + c_{17}(s))(t + c_{18}(s))} + \frac{2k(s)+1}{2}\pi
\end{aligned}
$$

with $c_{17/18}(s) = \frac{-c_{15} \pm \sqrt{c_{15}^2 - 4|c_{16}(s)|^2}}{-2\overline{i c}_{16}(s)}$. Since our integration path has angular extent less than π we can choose $k(s) \in \{-1, 0, 1\}$. We define two more integrals M

$$
\begin{aligned}
\int^y \frac{t^2}{(1-t^2)^3}\,dt &= \frac{y}{4(y^2-1)^2} + \frac{y}{8(y^2-1)} + \frac{1}{16}\ln\frac{y-1}{y+1} + \kappa_M \\
&=: M(y)
\end{aligned}
$$

and L

$$
\begin{aligned}
&\int^y \frac{t^2}{(1-t^2)^3}\ln(b+t)\,dt \\
= \quad & \frac{1}{16}\left(\frac{-b}{(b+1)^2}\ln(y-1) - \frac{b}{(b-1)^2}\ln(1+y) + \frac{2(b-y)}{(b^2-1)(y^2-1)}\right. \\
& + \left(\frac{2(b+y)(1+by)\left((b-y)^2+(by-1)^2\right)}{(b^2-1)^2(y^2-1)^2} + \ln\frac{(1-y)(1-b)}{(1+y)(1+b)}\right)\ln(b+y) \\
& \left. + Li_2\left(\frac{1-y}{1+b}\right) - Li_2\left(\frac{1+y}{1-b}\right)\right) + \kappa_L \\
=: \quad & L(b)(y)
\end{aligned}
$$

Here $\kappa_{L,M}$ are integration constants and

$$
Li_2(z) = \sum_1^\infty \frac{z^k}{k^2}
$$

is the dilogarithm (see [11]), closely related to the logarithm $\ln\frac{1}{1-z} = \sum_1^\infty \frac{z^k}{k}$. Its series representation is absolutely convergent in the unit disk. Using the functional relationship

$$
Li_2(z) = \frac{-\pi^2}{6} - \frac{\ln^2(-z)}{2} - Li_2(z^{-1})
$$

the dilogarithm is defined in the entire complex plane. Efficient code for the evaluation of the dilogarithm function can be found in most special function libraries, e.g. *fn* from the mail server at `netlib@research.att.com`.

With these functions in hand we are finally equipped to express the entire integral of Equation 4 in closed form

$$
\begin{aligned}
&\int_0^{c_2}\int_0^{c_0} \ln f(s,t)\,ds\,dt \\
= \quad & \left[\left(s+\frac{c_3}{2}\right)G(f(s,.))(t) + \frac{c_1}{2}H(f(s,.))(t)\right]\Bigg|_{s=0,t=0}^{s=c_0,t=c_2} - 2c_0c_2 \\
& + c_{14}c_{15}\Bigg[\pi(2k(s)+1)M(t) \\
& -i\left\{L(-c_{17}(s))(t) + L(-c_{18}(s))(t) - L(c_{17}(s))(t) - L(c_{18}(s))(t)\right\}\Bigg]\Bigg|_{s=0,t=\sqrt{\frac{c_{13}}{\bar{c}_{13}}}}^{s=c_0,t=\sqrt{\frac{c_{13}+c_2}{\bar{c}_{13}+c_2}}}
\end{aligned}
$$

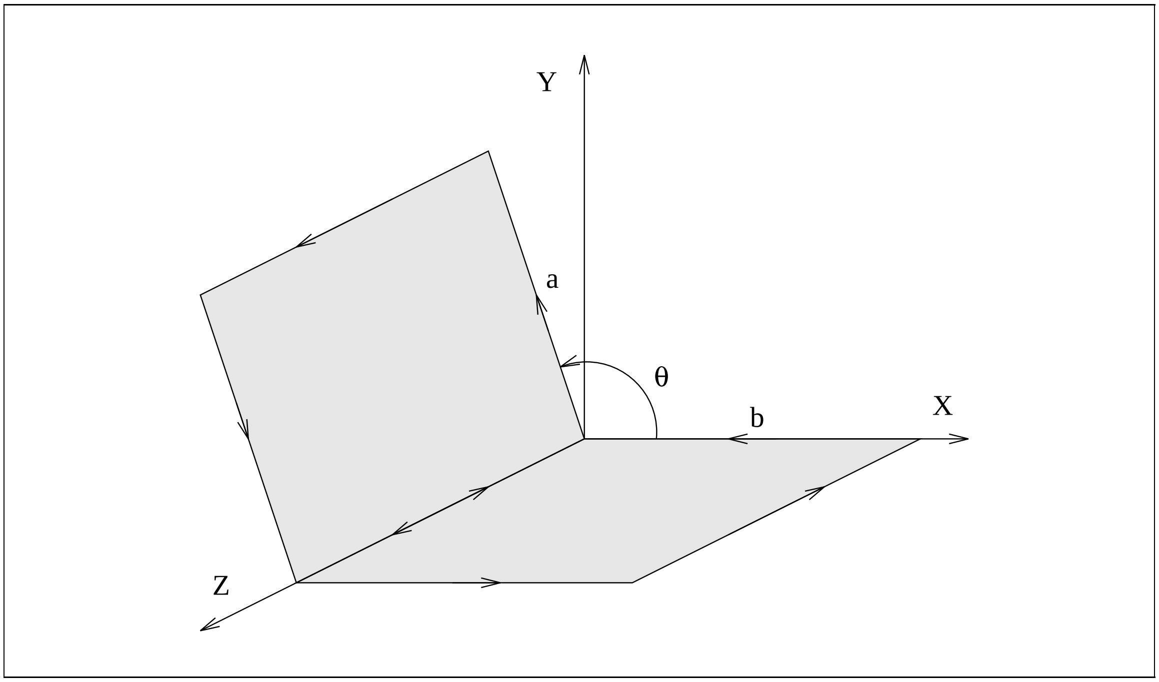

Figure 3: Geometry for two rectangles sharing a common edge with an enclosing angle of θ.

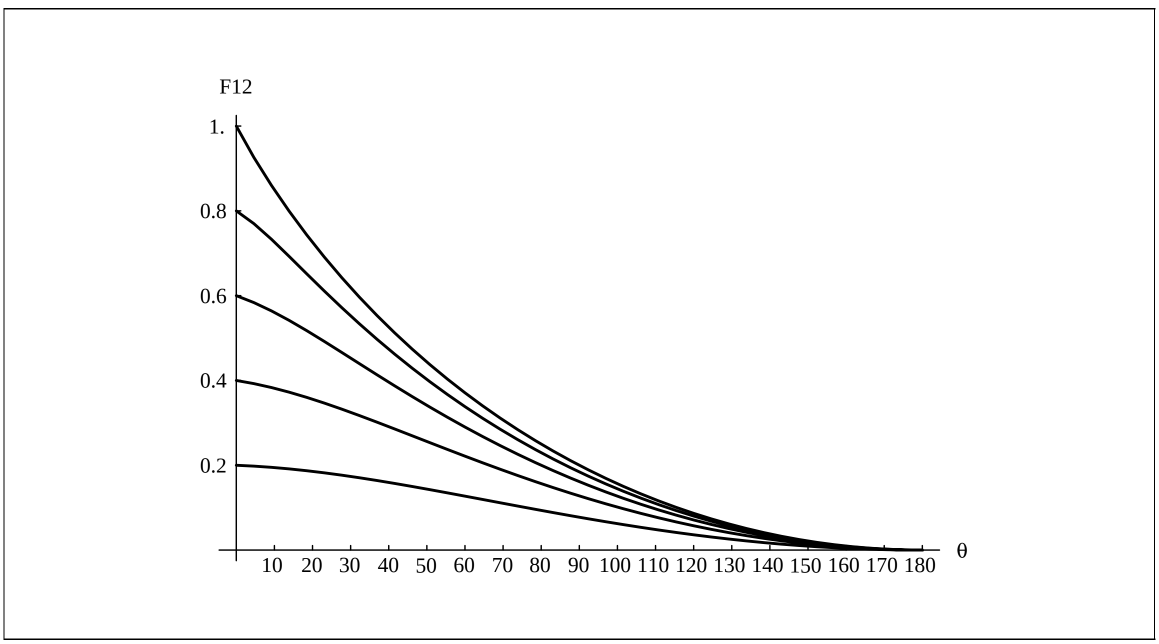

Figure 4: The form factor for the geometry in Figure 3 as a function of θ for edge ratios $l = \frac{a}{b}$ of .2, .4, .6, .8, and 1.0.

4 An example

The derivation above has been checked formally with a computer algebra system. However, in order to verify all constants and clarify any practical issues of the computation we have also implemented our formula. The latter point deserves some elaboration. Since we are required to go to the complex plane in order to symbolically integrate Equation 8, the fact that the involved functions are multivalued needs to be addressed explicitly. In particular, the branchcut of the complex logarithm needs to be placed away from the integration path. This does not place an undue burden on the code but requires careful implementation to avoid erroneous results.

As a simple example which requires the full power of our formula, we have computed the form factor between two equal width rectangles sharing an edge (see Figure 3) with an arbitrary angle θ between them. The literature gives a closed form solution for the case $\theta = \pi/2$ (e.g. in [16]). Figure 4 shows the form factor for this configuration as a function of θ for various aspect ratios $l = \frac{a}{b}$ and a length of 1 for the common edge. The results are in agreement with a quadrature of the original integral computed to sufficient accuracy.

5 Conclusion

We have shown that the form factor between two general polygons does admit a closed form solution. This solution is non-elementary in that it involves the dilogarithm function. The closed form can be used to advantage when checking other techniques or in situations in which standard numerical techniques are ill-conditioned, such as near singularities of the integrand.

There has been a long history of computing closed form expressions for form factors starting with Lambert in 1760. Since then there has been much progress in economizing the integrals involved, most notably the reduction from four dimensions to two with the contour integral method. The literature lists many special cases for which closed form solutions exist, but hitherto no solution had been given for general polygonal configurations. The present paper closes this gap.

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Appendix: List of all expressions

Before listing all the expressions to be computed we first define four auxiliary functions

$$\begin{aligned}
L(b)(y) &= \frac{1}{16}\left(\frac{-b}{(b+1)^2}\ln(y-1) - \frac{b}{(b-1)^2}\ln(1+y) + \frac{2(b-y)}{(b^2-1)(y^2-1)}\right. \\
&\quad + \left(\frac{2(b+y)(1+by)\left((b-y)^2+(by-1)^2\right)}{(b^2-1)^2(y^2-1)^2} + \ln\frac{(1-y)(1-b)}{(1+y)(1+b)}\right)\ln(b+y) \\
&\quad \left. + Li_2\left(\frac{1-y}{1+b}\right) - Li_2\left(\frac{1+y}{1-b}\right)\right) \\
M(y) &= \frac{y}{4(y^2-1)^2} + \frac{y}{8(y^2-1)} + \frac{1}{16}\ln\frac{y-1}{y+1} \\
G(q)(y) &= \frac{q'(y)}{2a}\ln q(y) - 2y + \frac{d}{a}\tan^{-1}\frac{q'(y)}{d} \\
H(q)(y) &= \left(\frac{y^2}{2} + \frac{c}{2a} - \frac{b^2}{4a^2}\right)\ln q(y) - \frac{y(ay-b)}{2a} - \frac{bd}{2a^2}\tan^{-1}\frac{q'(y)}{d}
\end{aligned}$$

where $q(x) = ax^2 + bx + c$ is some arbitrary quadratic polynomial and $d = \sqrt{4ac - b^2}$.

Given two edges we first compute the bi-quadratic form parameterizing the distance between the two edges as a function of s and t. Let E_i and E_j be parameterized by $\vec{x}_i(t) = \vec{p}_i + t\vec{d}_i$ and $\vec{x}_j(s) = \vec{p}_j + s\vec{d}_j$ with $\|\vec{d}_{i,j}\| = 1$, respectively. We have

$$\begin{aligned}
c_0 &= \|E_j\| \\
c_1 &= \vec{d}_i \cdot \vec{d}_j \\
c_2 &= \|E_i\| \\
c_3 &= -2\vec{d}_j \cdot (\vec{p}_i - \vec{p}_j)
\end{aligned}$$

$$
\begin{aligned}
c_4 &= 2\vec{d}_j \cdot (\vec{p}_i - \vec{p}_j) \\
c_5 &= \|\vec{p}_i - \vec{p}_j\|^2 \\
c_{10} &= 4 - c_1^2 \\
c_{11} &= 4c_4 - c_1 c_3 \\
c_{12} &= 4c_5 - c_3^2 \\
c_{13} &= \frac{c_{11} - \sqrt{c_{11}^2 - 4c_{10}c_{12}}}{2c_{10}} \\
c_{14} &= \frac{\sqrt{c_{11}^2 - 4c_{10}c_{12}}}{c_{10}} \\
c_{15} &= \sqrt{c_{10}}c_{14} \\
c_{16}(s) &= c_1 c_{13} - c_3 - 2s \\
c_{17}(s) &= \frac{-c_{15} + \sqrt{c_{15}^2 - 4|c_{16}(s)|^2}}{-2i\overline{c}_{16}(s)} \\
c_{18}(s) &= \frac{-c_{15} - \sqrt{c_{15}^2 - 4|c_{16}(s)|^2}}{-2i\overline{c}_{16}(s)}
\end{aligned}
$$

With these in hand we can compute the integral for a pair of edges

$$
\begin{aligned}
\mathcal{I}(E_i, E_j) &:= \int_0^{c_2} \int_0^{c_0} \ln f(s,t)\, ds\, dt \\
&= \left[\left(s + \frac{c_3}{2} \right) G(f(s,.))(t) + \frac{c_1}{2} H(f(s,.))(t) \right] \Bigg|_{s=0,t=0}^{s=c_0,t=c_2} - 2c_0 c_2 \\
&\quad + c_{14}c_{15} \Bigg[\pi(2k(s)+1)M(t) \\
&\quad -i \left\{ L(-c_{17}(s))(t) + L(-c_{18}(s))(t) - L(c_{17}(s))(t) - L(c_{18}(s))(t) \right\} \Bigg] \Bigg|_{s=0,t=\sqrt{\frac{c_{13}}{\bar{c}_{13}}}}^{s=c_0,t=\sqrt{\frac{c_{13}+c_2}{\bar{c}_{13}+c_2}}}
\end{aligned}
$$

in terms of which the form factor for two polygons is given by

$$
F_{P_1 P_2} = \frac{1}{4\pi A_{P_1}} \sum_{\substack{E_i \in \partial P_1 \\ E_j \in \partial P_2}} c_{1_{ij}}\, \mathcal{I}(E_i, E_j)
$$

Mathematica [19] code which implements the general case of our closed form for arbitrary polygons is available from the authors through `ps@cs.princeton.edu`.